Malaysian Journal of Mathematical Sciences 19(4): 1567–1583 (2025)
https://doi.org/10.47836/mjms.19.4.20

Malaysian Journal of Mathematical Sciences

Journal homepage: https://mjms.upm.edu.my

Integral Transform and its Multi-dimensional Analog with their Various Specialized Forms

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Received: 6 February 2025
Accepted: 3 July 2025

Abstract

This paper presents advanced techniques for transforming integrals involving special functions, focusing on extending integral transformations to multi−dimensional cases. Building on classical methods such as Euler and Laplace transformations, the study introduces new triple integral transformations, which are then generalized to higher dimensions. The key contribution of the research is the derivation of triple integral transformations for generalized hypergeometric functions and special polynomials like the Bessel and Rice polynomials. These transformations reduce complex integrals to more manageable forms, utilizing Gamma functions and hypergeometric series. The paper also introduces multi−dimensional analogs of these transformations, allowing the techniques to be applied to integrals with multiple variables common in mathematical physics and engineering problems. A notable extension is the incorporation of Fox's H−function, which broadens the applicability of the transformations to more generalized hypergeometric functions. This enables solutions to a wider range of mathematical problems, particularly complex integrals. Additionally, the paper presents various specialized forms of transformations, including generating relations, expansion, and summation formulas for well-known polynomials such as Gegenbauer and Bessel. The operational techniques offered a systematic framework for handling complex multidimensional integrals, making the methods applicable across theoretical and applied mathematics. This work provides a significant advancement in integral transform theory in mathematical analysis, physics, and engineering.

Keywords: triple integral transformation; generalized hypergeometric function; integral transforms; Gegenbauer polynomial.

1 Introduction

Researchers worldwide have been deeply engaged in studying partial differential equations and their solution methods because these equations play a crucial role in describing various real-world phenomena. Since developing mathematical models for complex systems with multiple variables is a challenging task, many mathematicians today focus on finding exact and approximate solutions to partial differential and integral equations using different techniques. Kilicman et al. [14] explore the use of the Sumudu transform for solving linear ordinary differential equations with constant coefficients and convolution terms. Over time, various methods have been applied to tackle these complex problems. One of the most effective approaches for solving such intricate real-world challenges is the triple integral transformation, which has shown great promise in mathematical analysis.

Recently, researchers have introduced various forms of triple integral transformations. These transformations have been successfully utilized in solving initial boundary value problems, including those related to heat and wave equations. Their unique properties and wide-ranging applications make them powerful tools for obtaining efficient solutions across various scientific fields [2]. Eltayeb and Kilicman [12] describe in their work the use of the double Laplace transform to solve non-homogeneous wave equations with non-constant coefficients. They apply the convolution theorem to convert constant coefficient wave equations into ones with variable coefficients. Wang and Xu [25] explain in their work that they introduced a novel triple mixed integral transformation method for solving initial-boundary value problems arising in partial differential equations. They studied the fundamental properties of the triple Laplace Sumudu transform and provided its application to basic functions.

Saadeh et al. [21] proposed a new triple integral transform called the Laplace Aboodh Sumudu transform by combining three classical transforms. This approach reduces linear partial differential equations involving three variables into solvable algebraic equations through its inverse form. Saadeh [20] emphasizes the introduction of the general triple transform, its properties, its relationships to other transforms, and its applications in the solution of partial differential equations. Kulmitra and Tiwari [15] explored an integral transformation related to an extended generalized Srivastava hypergeometric multivariable special function. Their study primarily focuses on extending the multivariable hypergeometric function and examining its Mellin and Inverse Mellin integral transformations.

Generalized hypergeometric functions are a class of special functions of a wide category of functions that generalize the classical hypergeometric functions, which have played a crucial role in the field of mathematics for more than two hundred years. These functions are characterized by their dependence on several parameters and can be articulated through series expansions. They can be expressed in terms of an infinite series and have properties such as analytic continuation and transformation formulas, making them versatile in mathematical analysis. Generalized hypergeometric functions find applications in various fields, including number theory and physics, particularly in solving differential equations and modeling phenomena in quantum mechanics and statistical mechanics. Their rich structure and interconnections with other special functions enhance their utility in different research domains. Rathie [19] presented a natural extension of Fox's H function, called the I function, and explored its convergence conditions, multiple series representations, fundamental properties, and notable special cases.

Rao and Shukla [18] summarize the key contributions of the paper, focusing on the generalization of the hypergeometric function, its integral representations, differentiation rules, and connections to other mathematical functions. Yan [27] contributes significantly to understanding generalized hypergeometric functions, establishing their uniqueness, connecting them to classical results, and exploring their asymptotic behavior, which may have far-reaching implications in various mathematical disciplines. Virchenko et al. [24] studied a new generalized hypergeometric function, its properties, and its connections to previous research is presented, emphasizing its relevance in mathematical and physical applications. Karp [13] highlights the paper's contributions to the field of mathematical analysis, particularly in the study of generalized hypergeometric functions, by providing new representations, establishing important properties, and deriving inequalities.

Integral transformations represent sophisticated mathematical methodologies that effectuate the conversion of functions into alternate domains, thereby enhancing the examination and resolution of diverse problems, especially within the realm of differential equations. Such transformations, including but not limited to the Fourier, Laplace, and Mellin transforms, are of critical significance in disciplines such as engineering, physics, and applied mathematics. Integral transforms are employed in boundary value problems, such as heat conduction and fluid flow, where they simplify complex equations into manageable forms. Currently, researchers prefer integral transforms over other mathematical methods for solving problems in science and engineering because of three key advantages:

1. They offer simplicity.
2. They provide exact results.
3. They eliminate the need for complex calculations.

Chan et al. [7] explain that Integral transformations are widely used in physics, engineering, and applied mathematics, particularly in solving the boundary value problems that arise in fracture mechanics. Crack problems in solid mechanics often lead to singular integral equations, where stress intensity factors need to be determined.

Bardzokas et al. [4] discuss that many of these problems can be transformed into solvable forms using integral transformations, including the Fourier, Laplace, and Mellin transforms, which simplify complex stress equations near crack tips. The generalized hypergeometric functions, Bessel polynomials, and Fox's H−function, which are central to this study, frequently appear in crack problems when modeling stress fields around fractures in elastic and viscoelastic materials. These functions help express solutions in closed form, making it easier to analyze stress concentration and predict crack propagation. Sitnik and Skoromnik [22] express that the multidimensional integral transforms developed in this paper provide a powerful framework for addressing such problems, especially in anisotropic materials where cracks interact with different boundary conditions. The results obtained here can be applied to stress analysis, thermal stress distribution, and wave propagation in cracked media, making these mathematical tools highly relevant to modern fracture mechanics.

Kumar et al. [16] introduced the "Rishi transform", a new integral transform aimed at finding exact solutions for the first kind Volterra integral equations. They first derived the transform for fundamental mathematical functions and examined its key properties. These properties enable its application to a wide range of equations, including ordinary, partial, delay, fractional, difference, integral, and integrodifferential equations. Nuruddeen et al. [17] discussed an integral transform-based decomposition method that combines a specific integral transform defined in the time domain. Aggarwal et al. [1] applied the Laplace transform to address population growth and decay problems. They also presented various applications to illustrate the effectiveness of the Laplace transform in solving such problems. Davies and Martin [8] discuss various numerical methods for inverting the Laplace transform, evaluating them on the basis of applicability, accuracy, efficiency, and ease of implementation. The classified methods are sample computation, exponential function expansion, Gaussian quadrature, bilinear transformation, and Fourier series. Asher [3] provided a concise overview of the Laplace and Inverse Laplace Transforms. He explored the concept of the Laplace transform, its applications, and its significance. In addition, he presented its definition along with a detailed discussion of its key properties, supported by comprehensive proofs. Durbin [11] introduced a method for the numerical inversion of the Laplace transform, serving as a natural extension of the approach developed by (Dubner and Abate method [10]).

The Gagenbauer polynomials form a class of orthogonal polynomials that generalize both Legendre and Chebyshev polynomials. Gegenbauer polynomials have applications across various mathematical fields, particularly in complex analysis and signal processing. Their unique properties facilitate the study of bi-univalent functions, integral equations, and signal reconstruction, showcasing their versatility. They are often used in mathematical analysis, particularly in solving problems in mathematical physics and approximation theory. De Micheli and Viano [9] proposed a straightforward and efficient algorithm for computing the Gegenbauer transform. This algorithm proves highly useful in developing spectral methods for the numerical solution of ordinary and partial differential equations relevant to various physical problems.

Xu [26] discussed the generalized Gegenbauer polynomials, which are orthogonal polynomials related to a specific weight function. It presents a new integral formula that connects these polynomials to h-harmonic polynomials and transforms Jacobi polynomials. Bavinck et al. [6] analyze the symmetric coupled processor model, a queueing system in which a server dynamically allocates its capacity between two customer streams, except when one stream is empty. The study explores the system's characteristics, including an exponential service time distribution and an arrival process governed by independent Poisson distributions. The work of Srivastava and Panda [23] is the basis for this study, which generates various results on generalized Bessel polynomials in addition to Rice polynomials and certain hypergeometric functions. We will utilize inverse Laplace transform methods and operational techniques derived from (9) throughout our subsequent discussions.

This work explores development methods for integral transformation techniques to resolve complicated multiple-variable integrals with special functions. New triple integral transformation techniques are developed because classical transforms produce insufficient results when handling higher-dimensional mathematical problems. These new transforms become the foundation for multi-dimensional versions. This research plans to develop ordered procedures for transformations of generalized hypergeometric functions, Bessel and Rice polynomials, plus the required implementation of Fox's H-function. The developed advanced operational techniques serve to simplify complex integrals while providing mathematical tools for solving partial differential equations, in addition to problems in physics and engineering.

For clarity, the structural framework of the present investigation is outlined in Figure 1, illustrating the logical sequence from theoretical formulation through multi-dimensional generalization to operational characterizations and concluding applications.

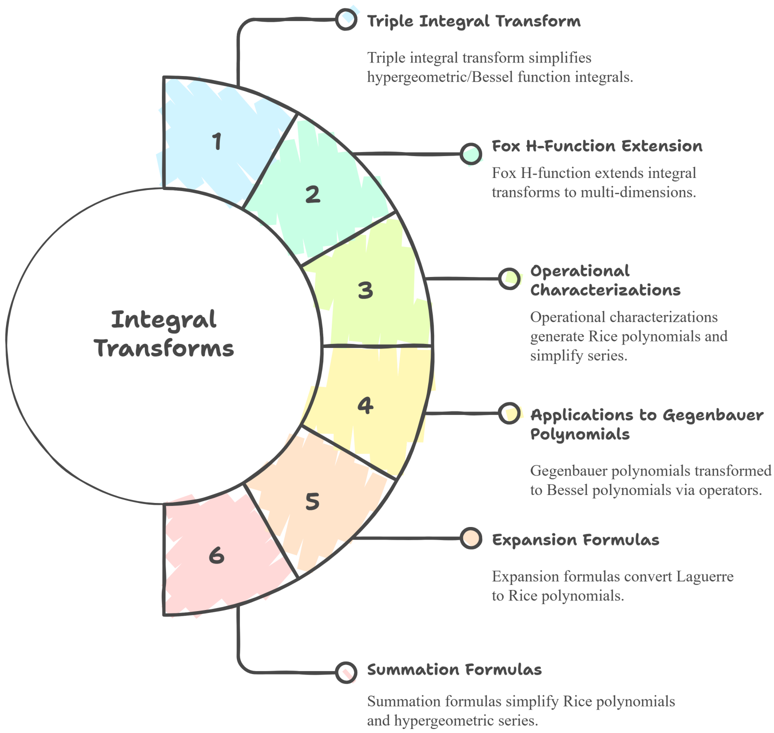

Figure 1: Flow structure of the article.

2 Main Result Derivation

Lemma 2.1. *Let ϕ be a function of three variables such that the integrals below converge, and let $\Re(\alpha) > 0$, $\Re(\beta) > 0$, and $\Re(\gamma) > 0$. Then, the following identity holds:*

$$\int_0^\infty \int_0^\infty \int_0^\infty \phi(u+v+w)u^{\alpha-1}v^{\beta-1}w^{\gamma-1}\,du\,dv\,dw = \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\alpha+\beta+\gamma)} \int_0^\infty \phi(z)z^{\alpha+\beta+\gamma-1}\,dz, \quad (1)$$

where the real part of $(\alpha, \beta, \gamma) > 0$ and (1) is the generalization of the Dirichlet integral often used to convert multivariable integrals into simpler one-variable integrals using gamma functions.

Proof. We begin by introducing a change of variables to simplify the triple integral. Let,

$$z = u+v+w, \quad u = zx, \quad v = zy, \quad w = z(1-x-y),$$

with $x, y \geq 0$ and $x+y \leq 1$. This change maps the domain of (u, v, w) to a 2D simplex in (x, y).

The Jacobian determinant of the transformation from (z, x, y) to (u, v, w). The Jacobian matrix is

$$J = \begin{bmatrix} \frac{\partial u}{\partial z} & \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial z} & \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial z} & \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{bmatrix} = \begin{bmatrix} x & z & 0 \\ y & 0 & z \\ 1-x-y & -z & -z \end{bmatrix}.$$

On computing the determinant of J,

$$\begin{aligned} |J| &= x \cdot \begin{vmatrix} 0 & z \\ -z & -z \end{vmatrix} - z \cdot \begin{vmatrix} y & z \\ 1-x-y & -z \end{vmatrix} + 0 \\ &= x(z^2) - z(zx - z) = xz^2 - xz^2 + z^2 = z^2. \end{aligned}$$

The Jacobian determinant becomes:

$$\left| \frac{\partial(u, v, w)}{\partial(z, x, y)} \right| = z^2.$$

Substituting into the original integral, we get,

$$\begin{aligned} &\int_0^\infty \phi(z) \int_0^1 \int_0^{1-x} (zx)^{\alpha-1}(zy)^{\beta-1}[z(1-x-y)]^{\gamma-1} z^2 \, dy \, dx \, dz \\ &= \int_0^\infty \phi(z) z^{\alpha+\beta+\gamma-1} \left[\int_0^1 \int_0^{1-x} x^{\alpha-1} y^{\beta-1} (1-x-y)^{\gamma-1} \, dy \, dx \right] dz. \end{aligned}$$

The inner double integral is a standard Beta integral over the triangle $x \geq 0$, $y \geq 0$, $x + y \leq 1$,

$$\int_0^1 \int_0^{1-x} x^{\alpha-1} y^{\beta-1} (1-x-y)^{\gamma-1} \, dy \, dx = \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\alpha+\beta+\gamma)}.$$

So, the full expression becomes:

$$\frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\alpha+\beta+\gamma)} \int_0^\infty \phi(z) z^{\alpha+\beta+\gamma-1} \, dz.$$

Hence, proved. □

Further, we start with the identity (1) involving a function ϕ over three variables. This identity facilitates the transformation of a triple integral into a single integral, which is instrumental in simplifying expressions involving special functions.

The term-by-term integration allows us to easily show that if $\Re(\alpha, \beta, \gamma) > 0$ and l, m, n, k are non-negative integer values, then the resulting hypergeometric series has a convergent region:

$$\begin{aligned} &\int_0^\infty \int_0^\infty \int_0^\infty \phi(u+v+w) u^{\alpha-1} v^{\beta-1} w^{\gamma-1} {}_pF_q \left[\begin{matrix} (a_p); \\ (b_q); \end{matrix} \; t u^{2l} v^{2m} w^{2n} (u+v+w)^{2k} \right] du \, dv \, dw \\ &= \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\alpha+\beta+\gamma)} \int_0^\infty \phi(z) z^{\alpha+\beta+\gamma-1} \; {}_{p+2l+2m+2n}F_{q+2l+2m+2n} \\ &\times \left[\begin{matrix} (a_p), \Delta(2l, \alpha), \Delta(2m, \beta), \Delta(2n, \gamma); \\ (b_q), \Delta(2l+2m+2n, \alpha+\beta+\gamma); \end{matrix} \; \zeta t z^{2(l+m+n+k)} \right], \end{aligned} \tag{2}$$

where

$$\zeta = \frac{(2l)^{2l}(2m)^{2m}(2n)^{2n}}{(2l+2m+2n)^{2(l+m+n)}},$$

and $\Delta(s,\alpha)$ stands for the set of s parameters $\frac{\alpha}{s}, \frac{\alpha+1}{s}, \ldots, \frac{\alpha+s-1}{s}$, (a_p) for $a_1,\ldots,a_p$ and so on.

In particular, if we let,

$$\phi(z) = z^{\sigma+1} K_\mu\left(\frac{1}{2}\lambda z\right) K_\nu\left(\frac{1}{2}\lambda z\right), \tag{3}$$

and by using formula,

$$\int_0^\infty x^{s-1} K_\mu(\alpha x) K_\nu(\alpha x)\, dx = \frac{2^{s-3}\alpha^{-s}\Gamma\left[\frac{1}{2}(s\pm\mu\pm\nu)\right]}{\Gamma(s)}, \tag{4}$$

to evaluate the right-hand side of (2), we shall arrive at the integral transformation,

$$\int_0^\infty\int_0^\infty\int_0^\infty u^{\alpha-1}v^{\beta-1}w^{\gamma-1}(u+v+w)^{\sigma+1} \times K_\mu\left[\frac{\lambda}{2}(u+v+w)\right] K_\nu\left[\frac{\lambda}{2}(u+v+w)\right]$$
$$\times\, {}_pF_q\left[\begin{matrix}(a_p);\\(b_q);\end{matrix}\; tu^{2l}v^{2m}w^{2n}(u+v+w)^{2k}\right] du\,dv\,dw$$
$$= \frac{2^{2(\alpha+\beta+\gamma+\sigma)-1}\Gamma\left(\frac{1}{2}(\alpha+\beta+\gamma\pm\mu\pm\nu+\sigma+1)\right)\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}{\lambda^{\alpha+\beta+\gamma+\sigma+1}\Gamma(\alpha+\beta+\gamma+\sigma+1)\Gamma(\alpha+\beta+\gamma)}$$
$$\times\, {}_{p+6l+6m+6n+4k}F_{q+4l+4m+4n+2k} \times$$
$$\left[\begin{matrix}(a_p),\Delta(2l,\alpha),\Delta(2m,\beta),\Delta(2n,\gamma), & \Delta(l+m+n+k),\frac{1}{2}(\alpha+\beta+\gamma\pm\mu\pm\nu+\sigma+1); \\ (b_q),\Delta(2l+2m+2n,\alpha+\beta+\gamma) & \Delta(2l+2m+2n+2k,\alpha+\beta+\gamma+\sigma+1);\end{matrix}\; \zeta\zeta' t\right], \tag{5}$$

where $\mathrm{Re}(\alpha+\beta+\gamma\pm\mu\pm v+\sigma+1)>0$, $\mathrm{Re}(\lambda)>0$ and $\zeta' = \left(\frac{2(l+m+n+k)}{\lambda}\right)^{2(l+m+n+k)}$ and for the sake of brevity, the pair of parameters like $(\alpha+\beta),(\alpha-\beta)$ has been abbreviated as $(\alpha\pm\beta)$ and with the product gamma $\Gamma(\alpha+\beta)\Gamma(\alpha-\beta)$ as $\Gamma(\alpha\pm\beta)$.

The results from Bhagchandani's work easily appear within expression (5). This follows from the known relationship [5]:

$$K_{\pm\frac{1}{2}}(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z}, \tag{6}$$

and from the application of the Gauss and Legendre multiplication formula:

$$\prod_{r=0}^{m-1}\Gamma\left(z+\frac{r}{m}\right) = (2\pi)^{\frac{1}{2}} m^{\frac{1}{2}-mz}\Gamma(mz), \quad \text{where} \quad m=2,3,4,\ldots.$$

When the value of m is equal to 2, this corresponds to Legendre's duplication formula:

$$\sqrt{\pi}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right), \tag{7}$$

and with the identity:

$$\Delta(2n,\lambda)=\Delta\left(n,\frac{\lambda}{2}\right),\Delta\left(n,\frac{\lambda+1}{2}\right), \tag{8}$$

where n is an integer ≥ 1, the integral transformation (5) with $\mu=\pm\frac{1}{2}$ and $\sigma, 2l, 2m, 2n, 2k$ replaced by μ, l, m, n, and k respectively,

$$\begin{aligned}\Omega^{\lambda,\mu,\nu}_{\alpha,\beta,\gamma,\sigma}\{\}=&\left[\frac{2^{2(\alpha+\beta+\gamma+\sigma)-1}\Gamma\left(\frac{1}{2}\left(\alpha+\beta+\gamma\pm\mu\pm\nu+\sigma+1\right)\right)\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}{\lambda^{\alpha+\beta+\gamma+\sigma+1}\Gamma(\alpha+\beta+\gamma+\sigma+1)\Gamma(\alpha+\beta+\gamma)}\right]^{-1}\\&\times\int_0^\infty\int_0^\infty\int_0^\infty u^{\alpha-1}v^{\beta-1}w^{\gamma-1}(u+v+w)^{\sigma+1}K_\mu\left(\frac{\lambda}{2}(u+v+w)\right)\\&K_\nu\left(\frac{\lambda}{2}(u+v+w)\right)\{\}\,du\,dv\,dw,\end{aligned} \tag{9}$$

and we wish to record a multi-dimensional analog of this operator given by:

$$\begin{aligned}\Omega^{\lambda,\mu,\nu}_{\alpha_1,\ldots,\alpha_n,\sigma}\{\}=&\left[\frac{2^{2(\alpha_1+\ldots+\alpha_n+\sigma)-1}\Gamma\left(\frac{1}{2}\left(\alpha_1+\ldots+\alpha_n\pm\mu\pm\nu+\sigma+1\right)\right)\Gamma(\alpha_1)\ldots\Gamma(\alpha_n)}{\lambda^{\alpha_1+\ldots+\alpha_n+\sigma+1}\Gamma\left(\alpha_1+\ldots+\alpha_n+\sigma+1\right)\Gamma\left(\alpha_1+\ldots+\alpha_n\right)}\right]^{-1}\\&\times\int_0^\infty\int_0^\infty\ldots\int_0^\infty u_1^{\alpha_1-1}\ldots u_n^{\alpha_n-1}(u_1+\ldots+u_n)^{\sigma+1}\\&K_\mu\left(\frac{\lambda}{2}(u_1+\ldots+u_n)\right)\times K_\nu\left(\frac{\lambda}{2}(u_1+\ldots+u_n)\right)\{\}\,du_1\ldots du_n.\end{aligned} \tag{10}$$

For $k=0$, (9) yields,

$$\begin{aligned}&\Omega^{\lambda,\mu,\nu}_{\alpha,\beta,\gamma,\sigma}\left\{{}_pF_q\left[\begin{array}{c}(a_p)\\(b_q)\end{array}\; tu^{2l}v^{2m}w^{2n}\right]\right\}\\&={}_{p+6l+6m+6n}F_{q+4l+4m+4n}\times\\&\left[\begin{array}{c}(a_p);\Delta(2l,\alpha),\Delta(2m,\beta),\Delta(2n,\gamma),\Delta(l+m+n,\frac{1}{2}(\alpha+\beta+\gamma\pm\mu\pm\nu+\sigma+1);\\(b_q);\Delta(2l+2m+2n,\alpha+\beta+\gamma),\Delta(2l+2m+2n,\alpha+\beta+\gamma+\sigma+1);\end{array}\; T\right],\end{aligned} \tag{11}$$

where

$$T=t\left(\frac{2l}{\lambda}\right)^{2l}\left(\frac{2m}{\lambda}\right)^{2m}\left(\frac{2n}{\lambda}\right)^{2n},$$

where l,m,n are non-negative integers and the parameters satisfy the conditions:
$\text{Re}(\alpha,\beta,\gamma,\lambda)>0$ and $\text{Re}(\alpha+\beta+\gamma\pm\mu\pm\nu+\sigma+1)>0$, it is assumed that both sides of the equation have a meaningful interpretation.

3 Generalization Using Fox's H-Function

This section presents an extension of the triple integral transformation involving the Fox's H-function, which generalizes various special functions. The formulation uses parameters $\alpha, \beta, \gamma, \sigma, \lambda, t$, sequences $(a_p, A_p), (b_q, B_q)$, and integer indices l, m, n, k, m', n', P, Q that define the structure and convergence of the transformation.

The H function, abbreviated for convenience as $H^{r,\delta}_{p,q}[z]$, is defined in a slightly modified form by:

$$H^{r,\delta}_{p,q}\left[z \left| \begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix} \right.\right] = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{r}\Gamma(b_j - B_j s)\prod_{j=1}^{\delta}\Gamma(1 - a_j + A_j s)}{\prod_{j=r+1}^{q}\Gamma(1 - b_j + B_j s)\prod_{j=\delta+1}^{p}\Gamma(a_j - A_j s)} z^s\, ds, \tag{12}$$

where $\{(a_p, A_p)\}$ stands for the sequence of p parameters $(a_1, A_1), (a_2, A_2), \ldots, (a_p, A_p)$, and so on. The integral in (12) converges if,

$$|\arg(z)| < \frac{1}{2}\mu'\pi, \tag{13}$$

where

$$\mu' = \sum_{j=1}^{\delta} A_j - \sum_{j=n+1}^{p} A_j + \sum_{j=1}^{r} B_j - \sum_{j=r+1}^{q} B_j > 0. \tag{14}$$

Expressed through the Mellin transform as:

$$F(s) = M\{f(z) : s\} = \int_0^\infty z^{s-1} f(z)\, dz. \tag{15}$$

It follows from (12) that,

$$M\{H^{r,\delta}_{p,q}(z) : s\} = \frac{\prod_{j=1}^{r}\Gamma(b_j + B_j s)\prod_{j=1}^{\delta}\Gamma(1 - a_j - A_j s)}{\prod_{j=r+1}^{q}\Gamma(1 - b_j - B_j s)\prod_{j=\delta+1}^{p}\Gamma(a_j + A_j s)}, \tag{16}$$

provided,

$$-\min_{1\le j\le r} \operatorname{Re}\left(\frac{b_j}{B_j}\right) < \operatorname{Re}(s) < \max_{1\le j\le \delta} \operatorname{Re}\left(\frac{1 - a_j}{A_j}\right).$$

Following the analysis used by Srivastava and Panda [23] in (1), we take,

$$\phi(z) = z^{\sigma} H^{m',n'}_{p,q}[\lambda z],$$

and appeal to (16), instead of (4), we thus derive the triple integral transformation where the integrand involves two Fox H-functions. Specifically, it generalizes the classical hypergeometric triple integral transformation to a more comprehensive framework involving generalized functions,

$$\int_0^\infty\int_0^\infty\int_0^\infty u^{\alpha-1}v^{\beta-1}w^{\gamma-1}(u+v+w)^\sigma H^{m',n'}_{p,q}\left[\lambda(u+v+w)\,\middle|\,\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right]$$
$$\times H^{M,N}_{P,Q}\left[-tu^lv^mw^n(u+v+w)^k\,\middle|\,\begin{matrix}(c_P,C_P)\\(d_Q,D_Q)\end{matrix}\right]du\,dv\,dw$$
$$=\lambda^{-\alpha-\beta-\gamma-\sigma}\times H^{M+n',N+m'+3}_{P+q+3,Q+p+1}\times \tag{17}$$
$$\left[-\frac{t}{\lambda^\theta}\,\middle|\,\begin{matrix}(1-\alpha,l),(1-\beta,m),(1-\gamma,n),(\lambda,\theta B_1),\dots,(\lambda_{m'},\theta B_{m'}),\\(c_P,C_P),(\lambda_{m'+1},\theta B_{m'+1}),\dots,(\lambda_q,\theta B_q)\times\\(\mu_{n'},\theta A_{n'}),(d_Q,D_Q),(1-\alpha-\beta-\gamma,l+m+n),\\(\mu_{n'+1},\theta A_{n'+1}),\dots,(\mu_p,\theta A_p)\end{matrix}\right],$$

where m',n',p,q and M,N,P,Q are integers satisfying the conditions that,

$$0\le m'\le q,\quad 0\le n'\le p,\quad 0\le M\le Q,\quad 0\le N\le P,$$

l,m,n,k along with $A_j\ (j=1,\dots,p)$, $B_j\ (j=1,\dots,q)$, $C_j\ (j=1,\dots,P)$, and $D_j\ (j=1,\dots,Q)$ are all positive values. The conditions of convergence, corresponding appropriately to (13) and (14), hold with $\mathrm{Re}(\alpha,\beta,\gamma,\lambda)>0$, and

$$-\min_{1\le j\le m'}\mathrm{Re}\left(\frac{b_j}{B_j}\right)<\mathrm{Re}(\alpha+\beta+\gamma+\sigma)<\max_{1\le j\le n'}\mathrm{Re}\left(\frac{1-a_j}{A_j}\right),$$
$$\begin{aligned}\theta&=l+m+n+k,\\\lambda_j&=1-b_j-(\alpha+\beta+\gamma+\sigma)B_j,\quad j=1,\dots,q,\\\mu_j&=1-a_j-(\alpha+\beta+\gamma+\sigma)A_j,\quad j=1,\dots,p.\end{aligned}\tag{18}$$

The mathematical relation (17) in analogy with (10) leads to this multi-dimensional integral transformation.

$$\int_0^\infty\int_0^\infty\dots\int_0^\infty u_1^{\alpha-1}\dots u_n^{\alpha_n-1}(u_1+\dots+u_n)^\sigma H^{m',n'}_{p,q}\left[\lambda(u_1+\dots+u_n)\,\middle|\,\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right]$$
$$\times H^{M,N}_{P,Q}\left[-tu_1^{\gamma_1}\dots u_n^{\gamma_n}(u_1+\dots+u_n)^\delta\,\middle|\,\begin{matrix}(c_P,C_P)\\(d_Q,D_Q)\end{matrix}\right]du_1\dots du_n$$
$$=\lambda^{-\sigma-\alpha_1-\dots-\alpha_n}H^{M+n',N+m'+n}_{P+q+n,Q+p+1}\times \tag{19}$$
$$\left[-\frac{t}{\lambda^\theta}\,\middle|\,\begin{matrix}(1-\alpha_1,\gamma_1),(1-\alpha_2,\gamma_2),\dots,(1-\alpha_n,\gamma_n),\\(\lambda_1,\theta B_1),\dots,(\lambda_{m'},\theta B_{m'}),(c_P,C_P),(\lambda_{m'+1},\theta B_{m'+1}),\dots,(\lambda_q,\theta B_q),\\(\mu'_n,\theta A'_n),(d_Q,D_Q),\\(1-\alpha_1-\dots-\alpha_n,\gamma_1+\dots+\gamma_n),(\mu'_{n'+1},\theta A'_{n'+1}),\dots,(\mu_p,\theta A_p)\end{matrix}\right],$$

where

$$-\min_{1\le j\le m'}\mathrm{Re}\left(\frac{b_j}{B_j}\right)<\mathrm{Re}(\alpha_1+\dots+\alpha_n+\sigma)<\max_{1\le j\le n'}\mathrm{Re}\left(\frac{1-a_j}{A_j}\right),$$
$$\begin{aligned}\theta&=\gamma_1+\dots+\gamma_n+\delta,\\\lambda_j&=1-b_j-(\alpha_1+\dots+\alpha_n+\sigma)B_j,\quad j=1,\dots,q,\\\mu_j&=1-a_j-(\alpha_1+\dots+\alpha_n+\sigma)A_j,\quad j=1,\dots,p.\end{aligned}\tag{20}$$

We demonstrate the use of (5) along with its multi-dimensional variant (10) as well as their distinct specialized versions.

4 Certain Operational Characterizations

This section delves into the operational properties of the operator $\Omega^{\lambda,\mu,\nu}_{\alpha,\beta,\gamma,\sigma}$, showcasing its action on a variety of special functions, including monomials, Laguerre polynomials, generalized hypergeometric functions, and Bessel polynomials. It establishes key identities, revealing deep connections to generalized Rice polynomials $H_n^{(\alpha,\beta)}$, and demonstrates their significance in both symbolic and applied mathematical analysis.

Observed that,

$$\Omega^{\lambda,\mu,\nu}_{\alpha,\beta,\gamma,\sigma}\{1\} = 1. \tag{21}$$

For $\lambda = 1$, we have

$$\Omega^{1,\mu,\nu}_{\alpha,\beta,\gamma,\sigma}\{\} \equiv \Omega^{\mu,\nu}_{\alpha,\beta,\gamma,\sigma}\{\}, \tag{22}$$

and

$$\Omega^{\mu,\nu}_{\alpha,\beta,\gamma,\sigma}\{u^l v^m w^n (u+v+w)^k\} = \frac{(\alpha)_l(\beta)_m(\gamma)_n\left[\frac{1}{2}(\alpha+\beta+\gamma\pm\mu\pm\nu+\sigma+1)\right]_{(l+m+n+k)/2}}{(\alpha+\beta+\gamma)_{l+m+n}(\alpha+\beta+\gamma+\sigma+1)_{l+m+n+k}}, \tag{23}$$

where the integers l, m, n and k are all greater than zero and satisfy the condition that their sum $l+m+n+k$ is an even number.

Additionally, we also have

$$\Omega^{1/2,1/2}_{\xi,1-\xi,\alpha,\beta}\{L_n^{p-1}(ux)\} = \frac{(p)_n}{(1+\alpha)_n} H_n^{(\alpha,\beta-n)}(\xi,p,x), \tag{24}$$

where $H_n^{(\alpha,\beta)}(\xi,p,x)$ represents the generalized Rice polynomial.

$$\Omega^{1/2,1/2}_{\xi,1-\xi,\alpha,\beta}\{(ux)^m L_n^{p-1}(ux)\} = \frac{x^m(\xi)_m(1+\alpha+\beta)_m(p)_n}{(1+\alpha)_m(1+\alpha+m)_n} H_n^{(\alpha+m,\beta-n)}(\xi+m,p,x), \tag{25}$$

$$\Omega^{\frac{1}{2},\frac{1}{2}}_{\frac{a}{2},\beta,\gamma,0}\left\{C_n^{\left(\frac{a-1}{2}\right)}\left(1+\frac{2ux}{b}\right)\right\} = \frac{(a-1)_n}{n!} y_n(a,b,x), \tag{26}$$

where $y_n(a,b,x)$ is the generalized Bessel polynomials:

$$y_n(a,b,x) = {}_2F_0\left(-n, a-1+n; -; -\frac{x}{b}\right),$$

$$\Omega^{\frac{1}{2},\frac{1}{2}}_{\alpha_1,\beta_1,\gamma_1,0}\left\{{}_1F_0\left[a;-;ux\right]\right\} = {}_2F_0\left[a,\alpha_1;-;x\right], \tag{27}$$

$$\Omega^{\frac{1}{2},\frac{1}{2}}_{\alpha_1,\beta_1,\gamma_1,0}\left\{{}_1F_1\left[a;b;ux\right]\right\} = {}_2F_1\left[\begin{matrix} a,\alpha_1; \\ b; \end{matrix}\; x\right], \tag{28}$$

$$\Omega^{\frac{1}{2},\frac{1}{2}}_{\xi,1-\xi,\alpha,\beta}\left\{{}_1F_1\left[a;b;ux\right]\right\} = {}_3F_2\left[\begin{matrix} a,1+\alpha+\beta,\xi; \\ b,1+\alpha; \end{matrix}\; x\right], \tag{29}$$

$$\Omega^{\frac{1}{2},\frac{1}{2}}_{\alpha,\beta-\gamma,\gamma,1}\left\{L_n^{(\alpha)}(ux)\right\} = H_n^{(\alpha,\beta-n)}(\alpha,\alpha+\beta,x). \tag{30}$$

5 Applications to Gegenbauer Polynomials

This section delves into the applications of the operator $\Omega^{\lambda,\mu,\nu}_{\alpha,\beta,\gamma,\sigma}$ in deriving generating relations for Gegenbauer polynomials $C_n^{(\nu)}(x)$. By employing operational methods and appropriate substitutions, we derive both classical and generalized generating functions for special functions, including generalized Bessel polynomials $y_n(a,b,x)$. These results establish connections with hypergeometric functions, expansion formulas, and identities involving modified Bessel functions $I_\nu(t)$, shedding light on their structural properties and applications.

The generating relation is given as:

$$(1-2xt+t^2)^{-\nu} = \sum_{n=0}^{\infty} C_n^{(\nu)}(x)t^n.$$

Rewriting it as:

$$\sum_{n=0}^{\infty} C_n^{(\nu)}(x)t^n = (1-t)^{-2\nu}{}_1F_0\left[\nu; -; \frac{-2t(1-x)}{(1-t)^2}\right]. \tag{31}$$

If in (31), we replace x by $\left(\frac{1+2ux}{b}\right)$, ν by $\left(\frac{a-1}{2}\right)$, and operate on both sides by $\Omega^{\frac{1}{2},\frac{1}{2}}_{\frac{a}{2},\beta,\gamma,0}$ using (26) and (27), we shall obtain a known (divergent) generating relation for the generalized Bessel polynomials, viz.,

$$\sum_{n=0}^{\infty} \frac{(a-1)_n}{n!} y_n(a,b,x)t^n \sim (1-t)^{-(a-1)}{}_2F_0\left[\frac{a-1}{2}, \frac{a}{2}; -; \frac{4xt}{b(1-t)^2}\right]. \tag{32}$$

The expansion formula gives:

$$\left[\frac{1}{2}(1-x)\right]^{\rho} = \frac{2^{2\lambda}}{\sqrt{\pi}}\Gamma(\lambda)\Gamma\left(\lambda+\rho+\frac{1}{2}\right)\sum_{n=0}^{\infty}\frac{(n+\lambda)(-\rho)_n}{\Gamma(n+2\lambda+\rho+1)}C_n^{(\lambda)}(x), \tag{33}$$

would yield another known result:

$$\left(\frac{-x}{b}\right)^{\rho} = \sum_{n=0}^{\infty}\frac{(2n+a-1)\Gamma(a+n-1)(-\rho)_n}{\Gamma(n+a+\rho)n!}y_n(a,b,x). \tag{34}$$

We also have three elegant relations, viz.,

$$2^n y_n\left(\frac{a}{2}-n+1, b, 2x\right) = \sum_{k=0}^{\frac{n}{2}} \frac{n!\left(\frac{a-1}{2}+n-2k\right)(a-1)_{n-2k}}{k!(n-2k)!\left(\frac{a-1}{2}\right)_{n-k+1}} y_{n-2k}(a,b,x), \tag{35}$$

$$\sum_{m=0}^{n}\frac{(-n)_m(a-1)_m y_m(a,b,x)}{m!(a/2-n)_m} = \frac{(-1)^m(a/2)_n}{(a/2-n)_n}{}_4F_2\left[\begin{matrix}-n, \frac{a-1}{2}, \frac{a}{2}+n, \frac{a}{2}; \\ \frac{a}{4}, \frac{a}{4}+\frac{1}{2};\end{matrix} \; -\frac{x}{b}\right], \tag{36}$$

and

$$\left(\frac{t}{2}\right)^{\frac{a-1}{2}} e^t \left(1-\frac{2xt}{b}\right)^{-\frac{a}{2}} = \Gamma\left(\frac{a-1}{2}\right)\sum_{n=0}^{\infty}\left(n+\frac{a-1}{2}\right)I_{n+\frac{a-1}{2}}(t)\frac{(a-1)_n}{n!}y_n(a,b,x). \tag{37}$$

The last formula admits a generalization, and we thus have

$$\sum_{n=0}^{\infty} \frac{\left(\frac{a+1}{2}\right)_k t^k}{k!\left(\frac{a}{2}+1\right)_k} y_k\left(\frac{a}{2}-n+1, b, 2x\right) = \frac{\Gamma(a+1)\, t^{-\frac{a-1}{2}}}{\left(\frac{a-1}{2}\right)\Gamma\left(\frac{a+1}{2}\right) 2^{\frac{a}{2}+1}},$$
$$\sum_{n=0}^{\infty}\left(n+\frac{a-1}{2}\right)\frac{(a-1)_n}{n!} y_n(a,b,x)\, I_{\frac{a}{4}-\frac{1}{2}+\frac{n}{2}}\left(\frac{t}{2}\right) I_{\frac{a}{4}+\frac{n}{2}}\left(\frac{t}{2}\right). \tag{38}$$

6 Expansion Formulas

This section investigates the application of operator expansion techniques, particularly through Laplace transform inversion, to derive expansions involving Laguerre polynomials and generalized Rice polynomials. Using these methods, we transform Laguerre polynomial expansions into corresponding results that involve generalized hypergeometric functions. This approach also facilitates the derivation of standard limiting cases, essential for generating multiplication formulas. The expansion formulas in (39) and (40) provide special cases of well-established expansion theorems for generalized hypergeometric functions, highlighting connections between hypergeometric series, polynomials, and their applications in both symbolic and applied analysis,

$$\begin{aligned}
&{}_{A+p+1}F_{B+p+1}\left[\begin{matrix}-n, (\rho_A), (\rho'_p);\\ c, (\sigma_B), (\sigma'_p);\end{matrix}\; xy\right]\\
&= \sum_{k=0}^{\infty}\frac{(-n)_k (b)_k [(\rho'_p)]_k (-y)^k}{k!(c)_k[(\sigma'_p)]_k} \times {}_{p+2}F_{p+1}\left[\begin{matrix}-n+k, b+k, (\rho'_p)+k;\\ c+k, (\sigma'_p)+k;\end{matrix}\; y\right]\\
&\times {}_{A+1}F_{B+1}\left[\begin{matrix}-k, (\rho_A);\\ b, (\sigma_A);\end{matrix}\; x\right],
\end{aligned} \tag{39}$$

$$\begin{aligned}
&{}_{A+1}F_{B+1}\left[\begin{matrix}-k, (a_A);\\ \beta, (b_B);\end{matrix}\; x\left(\frac{1+y}{2}\right)\right]\\
&= \sum_{r=0}^{k}\frac{(-k)_r(\alpha+\beta+2r)(\beta+r)_{k-r}(1+\alpha+\beta+2r)_{k-r}[(a_A)]_r}{(\beta)_k(\alpha+\beta+r)_{k+1}[(b_B)]_r}\\
&\times x^r P_r^{(\alpha,\beta)}(y)\, {}_{A+1}F_{B+1}\left[\begin{matrix}-k+r, (a_A)+r;\\ 1+\alpha+\beta+2r, (b_B)+r;\end{matrix}\; x\right].
\end{aligned} \tag{40}$$

These are the special cases of certain known expansion theorems for the generalized hypergeometric functions. It is assumed that,

$$[(a_A)]_k \text{ stands for } \sum_{j=1}^{A}(a_j)_k.$$

7 Summation Formulas

This section presents a collection of summation formulas involving generalized Rice polynomials $H_n^{(\alpha,\beta)}$ and hypergeometric functions. By applying transformation identities and operational methods, we extend classical results and establish new connections between different summation structures, including recurrence relations and operator-based transformations.

The following summation formulas are:

$$H_n^{(\alpha,\beta)}(\xi,p,x)=\frac{(1+\alpha)_n}{(p)_n}\sum_{r=0}^{n}\binom{n}{r}\frac{(p-r)_r}{(1+\alpha)_r}H_r^{(\alpha,\beta+n-r)}(\xi,p-r,x), \tag{41}$$

$$H_n^{(\alpha-n,\beta)}(\alpha,\alpha+\beta,x)=\sum_{r=0}^{n}(-1)^r\binom{n}{r}H_r^{(\alpha,\beta-r)}(\alpha,\alpha+\beta,x), \tag{42}$$

$$H_n^{(\alpha,\beta)}(\xi,p+q,x)=\frac{(1+\alpha)_n}{(p+q)_n}\sum_{r=0}^{n}\frac{(q)_r(p)_{n-r}}{r!(1+\alpha)_{n-r}}H_{n-r}^{(\alpha,\beta+r)}(\xi,p,x), \tag{43}$$

$$H_n^{(\alpha,\beta)}(\xi,p,x)=\sum_{m=0}^{n}\frac{(1+\alpha)_n(p-\beta-1)_m}{(p)_n m!}H_{n-m}^{(\beta,\alpha+m)}(\xi,1+\alpha,x). \tag{44}$$

All of these results can easily be extended to yield:

$$\frac{(p)_n}{n!}\,{}_{A+1}F_{B+1}\left[\begin{matrix}-n,(a_A);\\ p,(b_B);\end{matrix}\;x\right]=\sum_{r=0}^{n}\binom{n}{r}\frac{(p-r)_r}{r!}\,{}_{A+1}F_{B+1}\left[\begin{matrix}-r,(a_A);\\ p-r,(b_B);\end{matrix}\;x\right], \tag{45}$$

$$\frac{(\alpha-n)_n}{n!}\,{}_{A+1}F_{B+1}\left[\begin{matrix}-n,(a_A);\\ \alpha-n,(b_B);\end{matrix}\;x\right]=\sum_{r=0}^{n}(-1)^r\binom{n}{r}\frac{(\alpha)_r}{r!}\,{}_{A+1}F_{B+1}\left[\begin{matrix}-r,(a_A);\\ (\alpha,(b_B);\end{matrix}\;x\right], \tag{46}$$

$$\frac{(p+q)_n}{n!}\,{}_{A+1}F_{B+1}\left[\begin{matrix}-n,(a_A);\\ p+q,(b_B);\end{matrix}\;x\right]=\sum_{r=0}^{n}\frac{(q)_r(p)_{n-r}}{r!(n-r)!}\,{}_{A+1}F_{B+1}\left[\begin{matrix}-n+r,(a_A);\\ p,(b_B);\end{matrix}\;x\right], \tag{47}$$

$$\begin{aligned}&\frac{(p)_n}{n!}\,{}_{A+1}F_{B+1}\left[\begin{matrix}-n,(a_A);\\ p,(b_B);\end{matrix}\;x\right]\\ &\quad=\sum_{m=0}^{n}\frac{(p-\beta-1)_m(1+\beta)_{n-m}}{m!(n-m)!}\,{}_{A+1}F_{B+1}\left[\begin{matrix}-n+m,(a_A);\\ 1+\beta,(b_B);\end{matrix}\;x\right].\end{aligned} \tag{48}$$

All summation formulas (45)–(48) together with all results (41)–(44) come directly from standard recognized results. By using the operator $\Omega^{(1/2,1/2)}_{(\xi,1-\xi,\alpha,\beta+m)}$, we can deduce the following known relation:

$$\begin{aligned}&\sum_{r=0}^{n}\frac{(-n)_r(1+a)_r(1+\alpha)_m}{m!r!(1+b)_r}\,{}_5F_4\left[\begin{matrix}-m,1+\alpha+\beta+m,1+b+n,1+a+r,\xi;\\ 1+\alpha,1+\alpha,1+b+r,p;\end{matrix}\;x\right]\\ &\quad=\frac{(b-a)_n}{(1+b)_n}H_m^{(\alpha,\beta)}(\xi,p,x).\end{aligned} \tag{49}$$

Thus, on setting $\xi=p$, $x=1$, and applying Vandermonde's theorem, we obtain the sum,

$$\sum_{r=0}^{n}\frac{(-n)_r(b)_r}{r!(c)_r}\,{}_4F_3\left[\begin{matrix}-m,a+d+m,c+n,b+r;\\ a,b,c+r;\end{matrix}\;1\right]=\frac{(-)^m(c-b)_n(1+d)_m}{(c)_n(a)_n}. \tag{50}$$

The repeated application of operators presented in this paper through (49) results in an obvious hypergeometric sum,

$$\sum_{r=0}^{n}\frac{(-n)_r(\lambda)_r}{r!(\mu)_r}\,{}_{A+1}F_{B+1}\left[\begin{matrix}\lambda+r,(\rho_A);\\ \mu+r,(\sigma_B);\end{matrix}\;x\right]=\frac{(\mu-\lambda)_n}{(\mu)_n}\,{}_{A+1}F_{B+1}\left[\begin{matrix}\lambda,(\rho_A);\\ \mu+n,(\sigma_B);\end{matrix}\;x\right], \tag{51}$$

provided $A\le B$ (or $A=B+1$ and $|x|<1$).

8 Conclusions

The paper conducts a detailed investigation of integral transformations, stressing both their multi-dimensional counterparts alongside their distinct forms. The main accomplishments of this work is the creation of triple integral transformations for generalized hypergeometric functions, Bessel polynomials and Rice polynomials. These findings provide essential extensions to basic Euler and Laplace transforms by providing them in multiple dimensions which raises their capabilities for solving advanced integral operations.

The main result produced from this research establishes new integral transformations based on the H-function of Fox that add to numerous existing findings while expanding their capability to solve problems across diverse fields of mathematical physics and engineering. The proposed mathematical framework performs multidimensional integral computations through special functions and presents a step by step method for integration as shown in (5), (9) and (17).

This research deals with theoretical developments as well as implementation strategies that help solve boundary value problems and integral equations and differential equation systems. Section 4 offers operational characterizations which act as strong analytic tools for developing new solution methods that address practical challenges.

These research findings need further exploration because their scientists may extend them into fractional calculus as well as quantum mechanics and signal processing fields. The Gegenbauer polynomial relationships with summation rules create new paths for studying approximation theory along with orthogonal polynomials and numerical analysis. The developed foundation supports more conceptual research studies as well as scientific applications that span fields from engineering to natural sciences.

Acknowledgement The authors would like to acknowledge all those who contributed to the completion of this manuscript. We are grateful for the support, guidance, and constructive feedback received throughout the research and writing process. Appreciation is also extended to the editorial team and reviewers for their valuable comments and suggestions.

Conflicts of Interest: The authors declare no conflict of interest.

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